International Journal of Theoretical Physics, Vol. 30, No. 4, 1991

Energy Levels of λx^{2k} Anharmonic Oscillators Using the Quantum Normal Form

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Received June 21, 1990

The ground state and first few excited energy levels of the generalized anharmonic oscillator defined by the Hamiltonian $H=-d^2/dx^2+x^2+\lambda x^{2k}$ $(k=3, 4, \ldots)$ have been calculated by employing the method of quantum normal form, which is the quantum mechanical analogue of the classical Birkhoff–Gustavson normal form. The present energy eigenvalues are consistent with other tabulations of the energy levels.

1. INTRODUCTION

In recent years there has been a large and important literature on the methods for studying a well-known class of single-well quantum anharmonic oscillators. These one-body Schrödinger problems have played a particularly important role in recent years as model bosonic field theories which contain only one mode. This mode is generated by the usual harmonic oscillator creation operator $a^{\dagger}$. In this respect the anharmonic oscillators may be considered as the $(0+1)$-dimensional counterparts of more realistic quantum field theories in the physical world of $(3+1)$-dimensionality.

The present paper deals with the Schrödinger equation for the one-dimensional Hamiltonian operator

$$H=\tfrac{1}{2}(p^2+x^2)+\lambda x^{2k} \tag{1}$$

with $p=-i\,d/dx$, $k=2, 3, 4, \ldots$, which represents a $2k$-anharmonic oscillator. This problem has been attacked by a number of workers using different techniques (Arponen and Bishop, 1990; Biswas *et al.*, 1973; Hio *et al.*, 1976;

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0020-7748/91/0400-0487$06.50/0

Marziani, 1984; Taseli and Demirlap, 1988; Austin, 1984; Bhattacharya *et al.*, 1984; Banarjee, 1978; Fernandez *et al.*, 1985). We have employed the formalism of quantum normal form (QNF), which is the quantum mechanical analogue of the classical transformations to Birkhoff-Gustavson normal form (BGNF) (Birkhoff, 1927; Gustavson, 1966), to study the problem.

2. THEORY

We introduce the creation and annihilation operators in the basis set of harmonic oscillator wave functions,

$$a = 2^{-1/2}(x + ip) \tag{2}$$

$$a^{+} = 2^{-1/2}(x - ip) \tag{3}$$

$$p = -i\frac{d}{dx} \tag{4}$$

where the symbols have their usual meanings.

We have

$$[a, a^{+}] = 1 \tag{5}$$

$$a|n\rangle = n^{1/2}|n-1\rangle \tag{6}$$

$$a^{+}|n\rangle = (n+1)^{1/2}|n+1\rangle \tag{7}$$

where $|n\rangle$ represents the nth eigenket of the harmonic oscillator.

The Hamiltonian (1) has been discussed in the literature for various values of k. The case $k = 2$ has been studied independently by Ali (1985) and Eckhardt (1986) using the QNF, which has been used by Brajamani and Mazumdar (1988) to study the case $k = 3$ for $\lambda \ll 1$. However, they did not present the converged energy eigenvalues.

Transformations to normal form can be started from a Taylor expansion of the Hamiltonian around a point of equilibrium,

$$H = H_0 + \sum_{\mu} \lambda^{\mu} H_{\mu} \tag{8}$$

where

$$H_0 = a^{+}a + \tfrac{1}{2} \tag{9}$$

is the harmonic part and where H is a polynomial in a^+ and a, homogeneous of degree $\mu+2$. Obviously the quadratic part of H_0 is already in the normal form. A Lie transformation (Eckhardt 1986, 1988; Brajamani and Mazumdar, 1988) with a generator S_n and a "time variable" $\varepsilon=\lambda$ can be used to transform the increasing order of perturbation to normal form and we find that

$$H=H_0+\sum_{\mu=1}^{n-1}\lambda^{\mu}H_{\mu}+\lambda^{n}(H_n+[S_n, H_0])+O(\lambda^{n+1}) \tag{10}$$

Since lower-order terms are not affected by the transformation, S_n can be used to eliminate nonnormal terms in H_n. This requires solution of an equation

$$[S_n, H_0]+H_n=\text{normal} \tag{11}$$

As shown by Eckhardt (1988), equation (11) boils down to the fact that we have to find a self-adjoint operator S_n such that

$$[S_n, H_0]=-H_R \tag{12}$$

where H_R is the nonnormal part in H.

Using the ladder operators defined in equations (2) and (3), the quantum mechanical Hamiltonian operator of the one-dimensional x^{2k} oscillator is

$$H=\tfrac{1}{2}(p^2+x^2)+\frac{\lambda}{2^k}(a+a^+)^{2k} \tag{13}$$

In order to tackle the expression $(a+a^+)^{2k}$, the main problem arises from the fact that a and a^+ do not commute. But it is possible to express the bionomial expansion $(a^++a)^p$ through Newton binomials as (Duch, 1983)

$$(a^++a)^p=\sum_{m=0}^{[p/2]}(2m-1)!!\,{}^{p}c_{2m}(a^++a)_N^{p-2m} \tag{14}$$

$[p/2]$ is the integer part of $p/2$, and

$$(2m-1)!!=1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2m-1) \tag{15}$$

$(a^++a)_N^k$ is a Newton binomial, which is defined as

$$(a^++a)_N^k=\sum_{r=0}^{k}{}^{k}c_r(a^+)^{k-r}a^r \tag{16}$$

In the present case $p = 2k$. We have

$$(a+a^+)^{2k} = [(a^{+2k}+a^{2k}) + {}^{2k}c_2(a^{+(2k-2)}+a^{(2k-2)}) + 3!!\,{}^{2k}c_4(a^{+(2k-4)}+a^{(2k-4)}) + \cdots + (2k-1)!!] + [{}^{2k}c_1(a^{+(2k-1)}a + a^+a^{(2k-1)}) + {}^{2k}c_2(a^{+(2k-2)}a^2 + a^{+2}a^{(2k-2)}) + \cdots] + {}^{2k}c_2[{}^{(2k-2)}c_1(a^{+(2k-3)}a + a^+a^{(2k-3)}) + {}^{(2k-2)}c_2(a^{+(2k-4)}a^2 + a^{+2}a^{(2k-4)}) + \cdots] + 3!!\,{}^{2k}c_4[{}^{(2k-4)}c_1(a^{+(2k-5)}a + a^+a^{(2k-5)}) + {}^{(2k-4)}c_2(a^{+(2k-6)}a^2 + a^{+2}a^{(2k-6)}) + \cdots] + \cdots \tag{17}$$

Now using (13) and (17), the Hamiltonian for the sextic ($k=3$) and octic ($k=4$) anharmonic oscillators can be written as

$$H = H_0 + (H_N + H_R) \tag{18}$$

where H_N and H_R are, respectively, the normal and nonnormal parts of H:

$$H_N = \frac{\lambda}{8}(20a^{+3}a^3 + 90a^{+2}a^2 + 90a^+a + 15) \qquad (k=3) \tag{19}$$

$$H_N = \frac{\lambda}{16}(70a^{+4}a^4 + 560a^{+3}a^3 + 1260a^{+2}a^2 + 840a^+a + 105) \qquad (k=4) \tag{20}$$

$$H_R = \frac{\lambda}{8}[(a^{+6}+a^6) + 6(a^{+5}a + a^+a^5) + 15(a^{+4}a^2 + a^{+2}a^4) + 60(a^{+3}a + a^+a^3) + 15(a^{+4}+a^4) + 45(a^{+2}+a^2)] \qquad (k=3) \tag{21}$$

$$H_R = \frac{\lambda}{16}[(a^{+8}+a^8) + 8(a^{+7}a + a^+a^7) + 28(a^{+6}a^2 + a^{+2}a^6) + 56(a^{+5}a^3 + a^{+3}a^5) + 28(a^{+6}+a^6) + 168(a^{+5}a + a^+a^5) + 420(a^{+4}a^2 + a^{+2}a^4) + 210(a^{+4}+a^4) + 840(a^{+3}a + a^+a^3) + 420(a^{+2}+a^2)] \tag{22}$$

From equation (12) we find that to recast H in the normal form we have to find an operator S_n such that[3]

$$S_n = \frac{\lambda}{96}[2(a^{+6}-a^6) + 18(a^{+5}a - a^+a^5) + 90(a^{+4}a^2 - a^{+2}a^4) + 360(a^{+3}a - a^+a^3) + 45(a^{+4}-a^4) + 270(a^{+2}-a^2)] \qquad (k=3) \tag{23}$$

[3]It is to be noted that some errors crept into the expressions for S_n and E_n for the sextic oscillator ($k=3$) as reported by Brajamani and Mazumdar (1988).

and

$$S_n = \frac{\lambda}{384}[3(a^{+8} - a^8) + 32(a^{+7}a - a^+a^7) + 168(a^{+6}a^2 - a^{+2}a^6)$$
$$+ 672(a^{+5}a^3 - a^{+3}a^5) + 112(a^{+6} - a^6) + 1008(a^{+5}a - a^+a^5)$$
$$+ 5040(a^{+4}a^2 - a^{+2}a^4) + 1260(a^{+4} - a^4) + 10080(a^{+3}a - a^+a^3)$$
$$+ 5040(a^{+2} - a^2)] \qquad (k=4) \tag{24}$$

The crucial point is that once S_n is known, then it is a simple exercise to cast H in the normal form. Corresponding to equation (18), we obtain an expression for the eigenvalues from equation (10),

$$E_n = (n + \tfrac{1}{2}) + \frac{5\lambda}{8}(4n^3 + 6n^2 + 8n + 3) + \frac{\lambda^2}{192}(4716n^5 + 11{,}790n^4$$
$$+ 36{,}660n^3 + 43{,}200n^2 + 34{,}584n + 10{,}485) + \cdots \qquad (k=3) \tag{25}$$

$$E_n = (n + \tfrac{1}{2}) + \frac{35\lambda}{16}(2n^4 + 4n^3 + 10n^2 + 8n + 3)$$
$$+ \frac{\lambda^2}{192}(23{,}910n^7 + 83{,}685n^6 + 485{,}289n^5 + 1{,}004{,}010n^4$$
$$+ 2{,}057{,}055n^3 + 2{,}123{,}415n^2$$
$$+ 1{,}455{,}036n + 405{,}090) + \cdots \qquad (k=4) \tag{26}$$

3. RESULTS AND DISCUSSIONS

We find that the transformation to the normal form via a series of unitary transformations can be carried out to any desired order in λ. We have summed the normal form series following Ali *et al.* (1986). In Tables I and II we report the energy eigenvalues of sextic and octic anharmonic oscillators. The values reported are all consistent with other tabulations of the energy levels (Banarjee, 1978; Hioe *et al.*, 1976). This fact may be valuable when studying more realistic and consequently more complicated system.

ACKNOWLEDGMENTS

The authors thank Dr. S. P. Bhattacharya, Reader, Department of Theoretical Chemistry, Indian Association for the Cultivation of Science for his advice and suggestions and for providing the computational facility at the Horizon III computer system. Thanks are due to Prof. Dr. C. A. Singh, Dr. B. Bagchi, and Dr. Devraj for their help.

Table I. Energy Levels for the Sextic Anharmonic Oscillator

	E_0 $(n=0)$	E_1 $(n=1)$	E_2 $(n=2)$
0.0001	0.5002	1.501	2.505
0.001	0.5018	1.512	2.543
0.01	0.5154	1.595	2.794
0.1	0.5869	1.950	3.691
1	0.8048	2.875	5.772
10	1.282	4.756	9.807
100	2.192	8.254	17.18
	E_3 $(n=3)$	E_4 $(n=4)$	E_5 $(n=5)$
0.0001	3.512	4.524	5.542
0.001	3.604	4.702	5.842
0.01	4.132	5.606	7.209
0.1	5.774	8.147	10.78
1	9.325	13.41	17.98
10	16.04	23.24	31.30
100	28.22	40.99	55.27

Table II. Energy Levels for the Octic Anharmonic Oscillator

	E_0 $(n=0)$	E_1 $(n=1)$	E_2 $(n=2)$
0.0001	0.5006	1.506	2.524
0.001	0.5054	1.542	2.660
0.01	0.5321	1.705	3.140
0.1	0.6205	2.138	4.226
1	0.8207	3.000	6.211
10	1.191	4.500	9.532
100	1.816	6.967	14.91
	E_3 $(n=3)$	E_4 $(n=4)$	E_5 $(n=5)$
0.0001	3.571	4.590	5.678
0.001	3.904	5.023	6.412
0.01	4.881	6.297	8.325
0.1	6.869	8.881	11.99
1	10.33	13.37	18.25
10	16.02	20.75	28.45
100	25.17	32.60	44.78

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